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Correlations and coherence in the microwave instability

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Abstract

We expand upon the solvable model for the microwave instability in an electron storage ring as proposed in Wang (Phys. Rev. E 58 (1998) 984). The Vlasov–Maxwell equations are reduced to an integral eigenvalue problem similar to that which arises in the Karhunen–Loéve expansion theory for stochastic processes (Goodman, Statistical Optics, Wiley, New York, 1985). We derive first- and second-order correlation functions for the electron beam line density for the microwave instability. Using a set of coherent modes the correlation functions are diagonalized. A relationship between the first- and second-order correlation functions is obtained and it is shown to be similar to that of the Hanbury-Brown Twiss experiment (Born and Wolf, Principles of Optics, 6th Edition, Pergamon press, New York, 1980; Loudon, The Quantum Theory of Light, Clarendon press, Oxford, 1983). We obtain an expression for the Wigner distribution in terms of the first-order correlation function and relate this to the power in the electric field. We introduce an entropy-like quantity to characterize the coherence of the instability.

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1. Introduction

It has long been known [1,2] that the coherent microwave oscillation excited by a broadband impedance can be approximately characterized by a localized modulation function

$$\frac{\sin[(b+1/2)(\phi-\phi_c)]}{\sin[(\phi-\phi_c)/2]}$$

where ϕ_c is any real number, $-\pi \le \phi_c < \pi$, indicating the location inside the bunch where the coherent mode "c" is centered, and b is inversely proportional to the wakelength of the impedance. Based on the intuition suggested by this example, a solvable model for the microwave instability has recently been proposed [3].[1] The model consists of a bunched electron beam interacting with a constant impedance centered at the wave number $n = n_o$ and with a bandwidth of $\Delta n = 2b + 1$. The usual constraint

$$\lambda_o \ll l_W \ll \sigma \tag{1}$$

for microwave instability is understood, where σ is the bunch length, $\lambda_o = 2\pi/n_o$ is the wavelength of

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[1] We shall follow closely in this paper the convention and the notation of Ref. [3].

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the carrier wave and $l_W = 4\pi/(2b+1)$ is the wakelength characterizing the impedance; λ_o, l_W and σ are all in units of radians.

For our model of a storage ring with periodic boundary conditions, together with an impedance which is band-limited, the number of independent coherent modes is finite. A coherent mode of the beam interacting with the surrounding environment corresponds to an eigensolution of a linearized self-consistent Vlasov–Maxwell equation. The number of eigenfunctions is $2(2b+1)$ and they turn out to be given by

$$\exp(\pm i n_o \phi)\Gamma_\alpha(\phi) \quad (\alpha = 0, \pm 1, \pm 2, \ldots \pm b). \tag{2}$$

This corresponds to a carrier wave, $\exp(\pm i n_o \phi)$, modulated by the envelope function Γ_α, where

$$\Gamma_\alpha(\phi) = [2\pi(2b+1)]^{-1/2} \sum_{\nu=-b}^{b} \exp[i\nu(\phi - \phi_\alpha)] \tag{3a}$$

$$= [2\pi(2b+1)]^{-1/2} \frac{\sin[(b+1/2)(\phi - \phi_\alpha)]}{\sin[(\phi - \phi_\alpha)/2]} \tag{3b}$$

with

$$\phi_\alpha \equiv l_W \alpha/2. \tag{4}$$

The $2b+1$ envelope functions, Γ_α, are distributed symmetrically around the ring; the spacing between the center of the neighboring envelope functions is $l_W/2$, half the wakelength.

The modulation functions (3) are precisely the interpolating functions of the Shannon–Whittaker sampling theory generalized to the case of "periodic" band-limited functions [4]. However, the sampling theorem was not used in Ref. [3]. We shall make explicit use of the theorem in this paper, and by doing so the mathematical structure of the model becomes more transparent and many of the approximation schemes used in Ref. [3] are greatly simplified.

This paper is organized as follows: In Section 2, we recapitulate the results of the Ref. [3]. We also introduce in this section the sampling theorem and the Karhunen–Loéve problem for the band-limited shot noise $\mathscr{F}$. It turns out that the functions (3) above are also the eigenfunctions of the Karhunen–Loéve problem. In Section 3, we deal with the dynamic problem represented by the self-consistent Vlasov–Maxwell equations. We reduce the linearized Vlasov–Maxwell equations into a simple closed-loop integral equation for the band-limited component J of the electron beam current I. The closed-loop equation is a linear equation on a $2b+1$ dimensional modulation space $\mathscr{M}_b$. In Section 4, we show that the kernel of the closed-loop equation is precisely the linear operator in the Karhunen–Loéve problem. In other words, the kernel is the first-order correlation function $\langle \mathscr{F}\mathscr{F}^* \rangle$ of the band-limited shot noise. This is the reason for the simplicity of the model. The autocorrelation function of the radiation power is considered in Section 5. It was shown in Ref. [3] that the first-order correlation problems of this model reduce to the corresponding first-order correlation problems of the coherent modes. This remains true for the second-order correlation problems. We show that the Hanbury–Brown Twiss self-bunching relation for the chaotic photons [5] applies to the coherent states of this model. We also find a simple result for the correlation function of the total radiation power in the case of a flat but bunched beam. In Section 6, we show that the coherent-mode shot noise is a complex Gaussian random variable. In Section 7, we introduce a Wigner distribution and in Section 8, we define an entropy-like quantity to characterize the coherence of the instability.

2. Preliminaries

2.1. Impedance, passband and baseband

The model impedance used in Ref. [3] is assumed to be constant in a passband centered at a frequency $\pm n_o$ with a bandwidth $\Delta n = 2b+1$. For $n > 0$, the model impedance is defined by

$$Z_n = \begin{cases} \bar{Z} & \text{if } n_o - b \le n \le n_o + b \\ \bar{Z}^* & \text{if } -n_o - b \le n \le -n_o + b \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The impedance in the positive frequency and the negative frequency parts of the passbands are related by $Z_{-n} = Z_n^*$. A convenient quantity $\bar{U} \equiv -\mathrm{i}n_o \bar{Z}$ is also introduced.

The coherent motion of the beam is self-amplified through the interaction with the impedance which is present only in the passband. However, it is convenient to down convert the problem at hand mathematically from this passband to the associated baseband $\{n \mid n = 0, \pm 1, \pm 2, \ldots, \pm b\}$. We introduce a finite dimensional function space $\mathscr{M}_b$, the modulation space, corresponding to the baseband. $\mathscr{M}_b$ is defined to be the space spanned by the basis $\mathscr{B}_1 = \{\exp(\mathrm{i}\nu\phi) \mid \nu = 0, \pm 1, \pm 2, \ldots, \pm b\}$. An alternative basis $\mathscr{B}_2 = \{\Gamma_\alpha(\phi) \mid \alpha = 0, \pm 1, \pm 2, \ldots, \pm b\}$ of $\mathscr{M}_b$ turns out to be very useful, where the function Γ_α has been introduced in the introduction. Both $\mathscr{B}_1$ and $\mathscr{B}_2$ are complete orthogonal sets in the space $\mathscr{M}_b$. Therefore, taking into account the normalization constants, we have the following three integral kernel representations of the projection operator Q for the space $\mathscr{M}_b$:

$$Q(\phi - \phi') = \frac{1}{2\pi} \sum_{\nu=-b}^{b} \exp[\mathrm{i}\nu(\phi - \phi')] \tag{6a}$$

$$= \sum_{\alpha=-b}^{b} \Gamma_\alpha(\phi)\Gamma_\alpha(\phi') \tag{6b}$$

$$= \sqrt{\frac{2b+1}{2\pi}}\Gamma_0(\phi - \phi'). \tag{6c}$$

The projection operator Q will be used extensively below.

Whereas the member functions $\exp(\mathrm{i}\nu\phi)$ of $\mathscr{B}_1$ are spread throughout the entire ring, the member functions $\Gamma_\alpha(\phi)$ of $\mathscr{B}_2$ have the attractive property that they are concentrated in the neighborhood of ϕ_α; the exponential function $\exp(\mathrm{i}\nu\phi)$ represents a monochromatic oscillation, and Γ_α represents a modulation with finite bandwidth $2b + 1$. It turns out [1,2,6] that the function $\exp(\mathrm{i}\nu\phi)$ is not a eigenfunction of the microwave instability problem unless the electron beam is a coasting beam; however, the function Γ_α is the modulation function of an eigenfunction for any bunched beam in the microwave-coherent motion limit (1). We note that the role of $Q(\phi - \phi')$ inside the space $\mathscr{M}_b$ is similar to the role played by the periodic delta function $\delta(\phi - \phi')$ in the whole space of all periodic functions.

Having defined $\mathscr{M}_b$ to be the space of periodic functions on the baseband, let us discuss now the function space corresponding to the passband of Z. Define $\exp(\mathrm{i}n_o\phi) \otimes \mathscr{M}_b$ to be the space spanned by $\{\exp(\mathrm{i}n_o\phi + \mathrm{i}\nu\phi) \mid \nu = 0, \pm 1, \pm 2, \ldots, \pm b\}$. This is the function space corresponding to the positive frequency part of the passband. Similarly, the function space corresponding to the negative frequency part of the passband is given by $\exp(-\mathrm{i}n_o\phi) \otimes \mathscr{M}_b$ which is defined to be the space spanned by $\{\exp(-\mathrm{i}n_o\phi + \mathrm{i}\nu\phi) \mid \nu = 0, \pm 1, \pm 2, \ldots, \pm b\}$.

Let us end this part of the paper by introducing some mathematical theorems concerning the modulation space $\mathscr{M}_b$. Firstly, we introduce the sampling theorem [4,7] since it plays a crucial role in this paper.

Sampling Theorem: *Any function $g \in \mathscr{M}_b$ can be expanded in terms of the basis $\mathscr{B}_2$:*

$$g(\phi) = \sum_{\alpha=-b}^{b} \bar{g}_\alpha \Gamma_\alpha(\phi) \quad \text{with } \bar{g}_\alpha = \int_{-\pi}^{\pi} \mathrm{d}\phi g(\phi)\Gamma_\alpha(\phi). \tag{7}$$

The theorem states that

$$\bar{g}_\alpha = \sqrt{\frac{2\pi}{2b+1}} g_\alpha, \quad \text{where } g_\alpha \equiv g(\phi_\alpha). \tag{8}$$

Secondly, we draw attention to the following very useful integral

$$\int_{-\pi}^{\pi} \mathrm{d}\phi \Gamma_{\alpha_1}(\phi)\Gamma_{\alpha_2}(\phi) \ldots \Gamma_{\alpha_n}(\phi) = \left(\frac{2b+1}{2\pi}\right)^{(n-2)/2} \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} \ldots \delta_{\alpha_1\alpha_n}. \tag{9}$$

The Γ_α are the generalization of Shannon's sinc functions to our case of periodic boundary conditions.

2.2. Shot noise

The electron beam current distribution at time t is denoted by $I(\phi, t)$, and the microwave coherent motion is assumed to be initiated by the shot noise inherent in the electron beam. The initial current, I, consisting of N particles is given by

$$I(\phi, t=0) = eN\omega_{\rm o}F(\phi) \tag{10}$$

with the shot noise represented by

$$F(\phi) = \frac{1}{N}\sum_{j=1}^{N} \delta(\phi - \phi_j) \tag{11}$$

where $\delta(\phi)$ is a periodic delta function of period 2π, and the individual initial electron positions ϕ_j's are assumed to be independent and randomly distributed with a weight function $\rho(\phi_j)$. The function $\rho(\phi)$ has finite bandwidth proportional to $1/\sigma$, e.g. $\rho(\phi) \propto \exp(-\phi^2/2\sigma^2)$. It is very useful to define the projection of this function onto the space $\mathscr{M}_b$:

$$\rho^{(b)}(\phi) \equiv \int_{-\pi}^{\pi} {\rm d}\phi Q(\phi - \phi')\rho(\phi'). \tag{12}$$

From this definition and the assumption $l_{\rm W} \ll \sigma$ (see Introduction) we conclude that

$$\rho^{(b)}(\phi) \simeq \rho(\phi). \tag{13}$$

This is our primary approximation. In the rest of this paper, $\rho(\phi)$ will always be replaced by $\rho^{(b)}(\phi)$. Various approximation schemes used in Ref. [3] are consequences of the basic approximation in Eq. (13) combined with the sampling theorem.

The function $F(\phi)$ is a stochastic variable whose dependence on $\phi_1, \phi_2, \ldots, \phi_N$ is understood. The average of any stochastic quantity "A" which depends on $\phi_1, \phi_2, \ldots, \phi_N$ is given by

$$\langle A\rangle \equiv \prod_{j=1}^{N}\int {\rm d}\phi_j \rho(\phi_j)A \to \prod_{j=1}^{N}\int {\rm d}\phi_j \rho^{(b)}(\phi_j)A \tag{14}$$

where the approximation (13) has been used. Thus

$$\langle F(\phi)\rangle = \rho^{(b)}(\phi). \tag{15}$$

The function $\rho(\phi) \cong \rho^{(b)}(\phi)$ is normally referred to as the line density. This describes the electron beam of bunch length σ in equilibrium. As a consequence, the function

$$f(\phi) \equiv F(\phi) - \langle F(\phi)\rangle = F(\phi) - \rho^{(b)}(\phi) \tag{16}$$

describes the initial perturbation on the beam. This function has the desired property that

$$\langle f\rangle = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} {\rm d}\phi f(\phi) = 0. \tag{17}$$

The last equality amounts to the statement that no electrons are created by the initial beam perturbation.

Finally, we introduce the band-limited shot noise $\mathscr{F}(\phi)$ and $\mathscr{F}^*(\phi)$. Since the model impedance (5) is passband limited, only the component of the shot noise in this passband contribute to the startup and evolution of the microwave instability. It is convenient to map this component into the modulation space $\mathscr{M}_b$, ("down convert to the baseband"). The mapping of the positive frequency part of the passband is

$$\mathscr{F}(\phi) \equiv \int_{-\pi}^{\pi} {\rm d}\phi' \exp(-{\rm i}n_{\rm o}\phi')Q(\phi - \phi')F(\phi') \tag{18}$$

and the mapping of the negative frequency part is

$$\mathscr{F}^*(\phi) \equiv \int_{-\pi}^{\pi} {\rm d}\phi' \exp({\rm i}n_{\rm o}\phi')Q(\phi - \phi')F(\phi'). \tag{19}$$

Note that $\mathscr{F}$ and $\mathscr{F}^*$ are complex random variables since they depend implicitly on $\phi_1, \phi_2, \ldots, \phi_N$. Also, they are members of the modulation space $\mathscr{M}_b$ because Q is the projection operator for $\mathscr{M}_b$.

Since

$$\int_{-\pi}^{\pi} {\rm d}\phi' \, Q(\phi - \phi')\exp(-{\rm i}n_{\rm o}\phi')\rho^{(b)}(\phi') = 0 \tag{20}$$

the equilibrium part $\rho^{(b)}(\phi)$ of F does not contribute to the integral (18), and (18) is equivalent to

$$\mathscr{F}(\phi) = \int_{-\pi}^{\pi} {\rm d}\phi' \exp(-{\rm i}n_{\rm o}\phi')Q(\phi - \phi')f(\phi'). \tag{21}$$

In other words, only the perturbed part of the shot noise contributes to the band-limited shot noise. As mentioned earlier, we need only deal with $\mathscr{F}$ and $\mathscr{F}^*$ in the startup problem of the microwave instability.

2.3. First-order correlation of the shot noise and the Karhunen–Loève problem

The first-order correlation function of the shot noise is defined by

$$G^{(1)}(\phi,\phi') = \langle F(\phi)F(\phi')\rangle. \tag{22}$$

It is straight forward to verify, using Eqs. (11) and (14), that

$$G^{(1)}(\phi,\phi') = G_1^{(1)}(\phi,\phi') + G_2^{(1)}(\phi,\phi') \tag{23}$$

where

$$G_1^{(1)}(\phi,\phi') = \frac{1}{N}\rho^{(b)}(\phi)\delta(\phi-\phi') \tag{24}$$

$$G_2^{(1)}(\phi,\phi') = \frac{N-1}{N}\rho^{(b)}(\phi)\,\rho^{(b)}(\phi'). \tag{25}$$

Also, from Eqs. (16) and (14)

$$\langle f(\phi)f(\phi')\rangle = \frac{1}{N}\rho^{(b)}(\phi)\delta(\phi-\phi') - \frac{1}{N}\rho^{(b)}(\phi)\rho^{(b)}(\phi'). \tag{26}$$

We next project out the contribution to $G^{(1)}$ from the band-limited shot noise $\mathscr{F}$ and $\mathscr{F}^*$:

$$\langle\mathscr{F}(\phi)\mathscr{F}(\phi')\rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi''\,dp'''\exp[-in_o(\phi''+\phi''')] \times Q(\phi-\phi'')Q(\phi'-\phi''')G^{(1)}(\phi'',\phi''') \tag{27}$$

and

$$\langle\mathscr{F}(\phi)\mathscr{F}^*(\phi')\rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi''\,dp'''\exp[-in_o(\phi''-\phi''')] \times Q(\phi-\phi'')Q(\phi'-\phi''')\,G^{(1)}(\phi'',\phi'''). \tag{28}$$

Because of the relation (20), $G_2^{(1)}$ does not contribute to the integrals (27) and (28). As a consequence, for the treatment of the microwave instability, the following approximation is adequate:

$$G^{(1)}(\phi,\phi') \cong G_1^{(1)}(\phi,\phi') = \frac{1}{N}\rho^{(b)}(\phi)\delta(\phi-\phi'). \tag{29}$$

From Eq. (29) we conclude that the initial shot noise is characterized by a white noise stochastic process which by definition has a correlation length equal to zero [8].

The results of the integrations (27) and (28) are

$$\langle\mathscr{F}(\phi)\mathscr{F}(\phi')\rangle = \langle\mathscr{F}^*(\phi)\mathscr{F}^*(\phi')\rangle = 0 \tag{30}$$

and

$$\langle\mathscr{F}(\phi)\mathscr{F}^*(\phi')\rangle$$

$$= \frac{1}{N}\int_{-\pi}^{\pi} d\phi''Q(\phi-\phi'')Q(\phi'-\phi'')\rho^{(b)}(\phi'') \tag{31a}$$

$$= \frac{1}{N}\sqrt{\frac{2b+1}{2\pi}}\sum_{\alpha=-b}^{b}\bar{\rho}_\alpha^{(b)}\Gamma_\alpha(\phi)\Gamma_\alpha(\phi') \tag{31b}$$

$$= \frac{1}{N}\sum_{\alpha=-b}^{b}\rho_\alpha^{(b)}\Gamma_\alpha(\phi)\Gamma_\alpha(\phi') \tag{31c}$$

where the sampling theorem (8) is used to obtain Eq. (31c) from Eq. (31b).

Eq. (31) displays the band-limited first-order correlation function of the initial shot noise. Note that now the correlation length, which had been zero prior to the band-limiting operation, is now finite and equal to the wakelength l_W. This becomes transparent for a coasting beam where $\rho_\alpha^{(b)} = 1/2\pi$ and Eq. (31) reduces to $Q(\phi-\phi')$.

Eq. (31c) states that the correlation function $\langle\mathscr{F}\mathscr{F}^*\rangle$ treated as an operator on $\mathscr{M}_b$ is diagonalized by the base $\mathscr{B}_2$. In other words,

$$\int_{-\pi}^{\pi} d\phi'\langle\mathscr{F}(\phi)\mathscr{F}^*(\phi')\rangle\Gamma_\alpha(\phi') = \frac{1}{N}\rho_\alpha^{(b)}\,\Gamma_\alpha(\phi). \tag{32}$$

This is precisely the solution of the Karhunen–Loève problem [5,8] for the band-limited shot noise $\mathscr{F}$. Thus, an expansion like (7) is an Karhunen–Loève expansion with respect to the band-limited shot noise. The importance of Γ_α, from the point of view of the theory of random noise lies in the fact that the diagonalized expression (31c) embodies explicitly all the first-order correlation properties of the band-limited shot noise.

It is useful to introduce a Karhunen–Loéve expansion for the band-limited shot noise $\mathscr{F}$ itself

$$\mathscr{F}(\phi) = \sum_{\alpha=-b}^{b} \bar{\mathscr{F}}_\alpha \Gamma_\alpha(\phi). \tag{33}$$

We refer to the expansion coefficient $\bar{\mathscr{F}}_\alpha$ as the coherent-mode shot noise. It is given by

$$\bar{\mathscr{F}}_\alpha = \int \mathrm{d}\phi\, \mathscr{F}(\phi)\Gamma_\alpha(\phi) \tag{34a}$$

$$= \int \mathrm{d}\phi\, \Gamma_\alpha(\phi) \exp(-\mathrm{i}n_\mathrm{o}\phi)\, F(\phi). \tag{34b}$$

Eq. (34b) together with Eqs. (30) and (31) give the following important relations:

$$\langle \bar{\mathscr{F}}_\alpha \bar{\mathscr{F}}_\beta \rangle = 0$$

$$\langle \bar{\mathscr{F}}_\alpha^* \bar{\mathscr{F}}_\beta^* \rangle = 0$$

$$\langle \bar{\mathscr{F}}_\alpha \bar{\mathscr{F}}_\beta^* \rangle = \mathscr{G}_\alpha^{(1)} \delta_{\alpha\beta} \tag{35}$$

where

$$\mathscr{G}_\alpha^{(1)} \equiv \langle |\bar{\mathscr{F}}_\alpha|^2 \rangle = \rho_\alpha^{(b)}/N. \tag{36}$$

We have the critical result that the coherent mode shot noise $\bar{\mathscr{F}}_\alpha$ are statistically uncorrelated. The band-limited version of the first-order correlation function of the initial shot noise can then be written as

$$\mathscr{G}^{(1)}(\phi,\phi') = \langle \mathscr{F}(\phi)\mathscr{F}^*(\phi') \rangle \tag{37}$$

$$= \sum_{\alpha=-b}^{b} \mathscr{G}_\alpha^{(1)} \Gamma_\alpha(\phi)\Gamma_\alpha(\phi'). \tag{38}$$

Let us end this section by considering the power associated with the band-limited shot noise $\mathscr{F}$. The quantity

$$P_{\mathscr{F}}(\phi) \equiv \langle |\mathscr{F}(\phi)|^2 \rangle \tag{39}$$

is proportional to the power density associated with $\mathscr{F}$. This equation together with Eq. (31) gives

$$P_{\mathscr{F}}(\phi) = \frac{1}{N} \sum_{\alpha=-b}^{b} \rho_\alpha^{(b)} \Gamma_\alpha^2(\phi). \tag{40}$$

The corresponding total power is

$$P_{\mathscr{F}} = \int_{-\pi}^{\pi} \mathrm{d}\phi\, P_{\mathscr{F}}(\phi) \tag{41}$$

$$= \sum_{\alpha=-b}^{b} \mathscr{G}_\alpha^{(1)} \tag{42}$$

where $\mathscr{G}_\alpha^{(1)}$ is given by Eq. (36). Thus, $\mathscr{G}_\alpha^{(1)}$ is the band-limited shot noise power in the localized mode α, and the normalized quantity

$$\mathscr{G}_\alpha^{(1)} \Big/ \sum_\beta \mathscr{G}_\beta^{(1)} = \rho_\alpha^{(b)} \Big/ \sum_\beta \rho_\beta^{(b)} \tag{43}$$

can be considered to be the relative probability that the α mode of the band-limited shot noise is excited. We will expand on this idea further in a later section concerning entropy.

3. Simplification of the Vlasov–Maxwell equations

In Ref. [3] the linearized self-consistent Vlasov–Maxwell equations for the microwave instability were reduced to a Fredholm integral equation for the perturbed electron current distribution

$$\ddot{I}(\phi,t) = \kappa\, \rho^{(b)}(\phi) \int_{-\pi}^{\pi} \mathrm{d}\phi' \left[\bar{U} \exp(\mathrm{i}n_\mathrm{o}(\phi-\phi') + \mathrm{c.c.}\right]$$
$$\times Q(\phi-\phi')\, I(\phi',t) \tag{44}$$

where $\kappa = e\alpha\omega_\mathrm{o}^2 I_\mathrm{av}/E_0$ and $I_{av} = eN\omega_\mathrm{o}/2\pi$. The initial condition is $\dot{I}(\phi,0) = 0$ with $\dot{I}(\phi,t) \equiv \partial I(\phi,t)/\partial t$.

To proceed, we introduce some important simplifying notation,

$$I(\phi,t) = \exp(\mathrm{i}n_\mathrm{o}\phi)\, J(\phi,t)$$
$$+ \exp(-\mathrm{i}n_\mathrm{o}\phi)\, J^*(\phi,t) + \breve{I}(\phi,t) \tag{45}$$

where J, and $J^* \in \mathscr{M}_b$ are band-limited functions and $\breve{I}$ represents that portion of I orthogonal to the spaces $\exp(\mathrm{i}n_o\phi)\otimes\mathscr{M}_b$ and $\exp(-\mathrm{i}n_o\phi)\otimes\mathscr{M}_b$. Clearly,

$$J(\phi,t)=\int_{-\pi}^{\pi}\mathrm{d}\phi'\,\mathrm{e}^{-\mathrm{i}n_o\phi'}Q(\phi-\phi')\,I(\phi',t). \tag{46}$$

Let us obtain some expressions for J and $\breve{I}$. Substituting Eq. (45) into Eq. (44) and doing the integrals, we obtain

$$\ddot{I}(\phi,t)=\kappa\,\rho^{(b)}(\phi)\,[\bar{U}\exp(\mathrm{i}n_o\phi)J(\phi,t)+\text{c.c.}].$$

Using Eq. (46),we can project J from I on the left-hand side of above equation. The result is

$$\ddot{J}(\phi,t)=\kappa\bar{U}\int_{-\pi}^{\pi}\mathrm{d}\phi'\,Q(\phi-\phi')\rho^{(b)}(\phi')J(\phi',t). \tag{47}$$

It is also straightforward to obtain

$$\ddot{\breve{I}}(\phi,t)=[\kappa\,\bar{U}\exp(\mathrm{i}n_o\phi)\,\rho^{(b)}(\phi)\,J(\phi,t)-\exp(\mathrm{i}n_o\phi)\ddot{J}(\phi,t)]+\text{c.c.} \tag{48}$$

In Eq. (47), the second time derivative of J is expressed in terms of J alone, $\breve{I}$ is not involved in this equation. This equation will be referred to as the "self-consistent band-limited closed-loop equation", or the "closed-loop equation" in short. On the other hand, Eq. (48) expresses the second time derivative of $\breve{I}$ in terms of J; there is no dependence on $\breve{I}$ on the right-hand side. Once J is known, $\breve{I}$ can be determined from Eq. (48) through direct integrations. We refer to this equation as the "mode-coupling equation". The need for discussing the mode-coupling equation will not rise in the rest of this paper.

We reduced in this section the linearized Vlasov–Maxwell equations to a simple "closed-loop equation" in the space $\mathscr{M}_b$. We discuss this simplified equation in the next section.

4. Diagonalization of the self-consistent closed-loop equation

We solve the closed-loop equation (47) in this section. Particularly, we relate the kernel of this equation to the diagonalized correlation function (31) of the band-limited shot noise $\mathscr{F}$. In doing so, the inherent reason of the simplicity of this model becomes apparent.

Since $J\in\mathscr{M}_b$, and Q is the projection operator for the space $\mathscr{M}_b$, we have $J=QJ$, or equivalently

$$J(\phi',t)=\int_{-\pi}^{\pi}\mathrm{d}\phi''Q(\phi'-\phi'')J(\phi'',t). \tag{49}$$

Substituting this equation into Eq. (47), we obtain

$$\ddot{J}(\phi,t)=\kappa\bar{U}\int_{-\pi}^{\pi}\mathrm{d}\phi''\int_{-\pi}^{\pi}\mathrm{d}\phi'\,Q(\phi-\phi')Q(\phi'-\phi'')\rho^{(b)}(\phi')J(\phi'',t). \tag{50}$$

Using Eq. (31a) and $Q(\phi'-\phi'')=Q(\phi''-\phi')$ on the last equation, we obtain

$$\ddot{J}(\phi,t)=\kappa\,\bar{U}N\int_{-\pi}^{\pi}\mathrm{d}\phi''\langle\mathscr{F}(\phi)\mathscr{F}^*(\phi'')\rangle J(\phi'',t). \tag{51}$$

The kernel of the closed-loop equation turns out to be the first-order correlation function of the band-limited shot noise $\mathscr{F}$. Therefore, the solution to the equation can be obtained just by Karhunen–Loéve expanding J with respect to this noise. This is the reason why the present model which was first discussed in Ref. [3] was so simply "solvable".

Eq. (51) can be written as

$$\ddot{J}(\phi,t)=\kappa\bar{U}\int_{-\pi}^{\pi}\mathrm{d}\phi'\sum_{\alpha=-b}^{b}\rho_\alpha^{(b)}\Gamma_\alpha(\phi)\,\Gamma_\alpha(\phi')J(\phi',t). \tag{52}$$

We first solve this equation as an eigenvalue problem. To this end, we introduce a Karhunen–Loéve expansion for J:

$$J(\phi,t)=\sum_{\alpha=-b}^{b}\bar{J}_\alpha(t)\Gamma_\alpha(\phi) \tag{53}$$

where

$$\bar{J}_\alpha(t)=\int_{-\pi}^{\pi}\mathrm{d}\phi\,J(\phi,t)\Gamma_\alpha(\phi) \tag{54a}$$

$$=\int_{-\pi}^{\pi}\mathrm{d}\phi\,\Gamma_\alpha(\phi)\exp(-\mathrm{i}n_o\phi)I(\phi,t) \tag{54b}$$

and substitute Eq. (53) into Eq. (52). The result is

$$\ddot{\bar{J}}_\alpha=\kappa\bar{U}\rho_\alpha^{(b)}\,\bar{J}_\alpha. \tag{55}$$

Thus the eigensolution is

$$\bar{J}_\alpha(t) \sim \exp(\pm \mathrm{i}\Omega_\alpha t) \quad (56)$$

with

$$\Omega_\alpha^2 = -\kappa \bar{U} \rho_\alpha^{(b)}. \quad (57)$$

In order to solve the initial value problem, we combine the above results with the initial condition $\dot{I}(\phi,0) = 0$. The result is[2]

$$\bar{J}_\alpha(t) = \bar{J}_\alpha(0)\cos \Omega_\alpha t. \quad (58)$$

The initial value $\bar{J}_\alpha(0)$ is related to the coherent-mode shot noise $\bar{\mathscr{F}}_\alpha$. Using the transformations (34b) and (54b) together with the relation (10), we have

$$\bar{J}_\alpha(0) = eN\omega_o \bar{\mathscr{F}}_\alpha. \quad (59)$$

We have thus found the solution to the initial value problem.

Using the results of Eqs. (35), (58), and (59) it can be shown that the first-order correlation function for the band-limited current is given by

$$\mathscr{J}^{(1)}(\phi,\phi',t) = \langle J(\phi,t)J^*(\phi',t)\rangle \quad (60)$$

$$= (eN\omega_0)^2 \sum_{\alpha=-b}^{b} \mathscr{G}_\alpha^{(1)} |\cos \Omega_\alpha t|^2 \Gamma_\alpha(\phi)\Gamma_\alpha(\phi'). \quad (61)$$

The above results have been used in Ref. [3] to calculate the radiation power resulting from the microwave instability when the passband of the impedance is above the beam pipe-cutoff frequency. Instead of repeating the detailed discussion on these subjects, let us list here some of the radiation-power-related results needed for the following discussions.

The total power is

$$P_{\mathrm{tot}}(t) = \int_{-\pi}^{\pi} \mathrm{d}\phi\, P(\phi,t) \quad (62)$$

[2] For the quantity $\bar{J}_\alpha(0)$ in this paper, the notation $I_\alpha^{(0)}$ was used in Ref. [2].

where

$$P(\phi,t) = \frac{(eN\omega_o)^2}{2\pi} \sum_{\alpha,\beta} [\bar{Z}\bar{\mathscr{F}}_\alpha \bar{\mathscr{F}}_\beta^* \cos \Omega_\alpha t \cos \Omega_\beta^* t + \mathrm{c.c.}]\Gamma_\alpha(\phi)\Gamma_\beta(\phi). \quad (63)$$

Doing the integration of (62), we obtain

$$P_{\mathrm{tot}}(t) = \sum_\alpha P_\alpha(t) \quad (64)$$

where

$$P_\alpha(t) = P_\alpha^{(0)} |\cos \Omega_a t|^2 \quad (65)$$

$$P_\alpha^{(0)} = I_{\mathrm{av}}^2\, 4\pi \bar{R} |\bar{\mathscr{F}}_\alpha|^2, \quad (66)$$

the average current $I_{\mathrm{av}} \equiv eN\omega_o/2\pi$, and $\bar{R}$ is the resistive part of $\bar{Z}$. Note that the above P's are all random variables. From Eqs. (65) and (66)

$$\langle P_\alpha(t)\rangle = \langle P_\alpha^{(0)}\rangle |\cos \Omega_\alpha t|^2 \quad (67)$$

and

$$\langle P_\alpha^{(0)}\rangle = I_{\mathrm{av}}^2\, 4\pi \bar{R} \mathscr{G}_\alpha^{(1)} \quad (68)$$

where the coherent-mode shot noise power $\mathscr{G}_\alpha^{(1)}$ has been given in Eq. (36).

For completeness we include the ensemble averaged quantities,

$$\langle P(\phi,t)\rangle = \bar{R}\frac{(eN\omega_o)^2}{\pi N} \sum_{\alpha=-b}^{b} \rho_\alpha^{(b)} |\cos \Omega_\alpha t|^2 \Gamma_\alpha^2(\phi) \quad (69)$$

$$\langle P_{\mathrm{tot}}(t)\rangle = \bar{R}\frac{(eN\omega_o)^2}{\pi N} \sum_{\alpha=-b}^{b} \rho_\alpha^{(b)} |\cos \Omega_\alpha t|^2. \quad (70)$$

5. The correlation function of the radiation power

We treat in this section the autocorrelation function of the radiation power. Let us first relate the correlation function of the total radiation power P_{tot} to that of P_α. From Eqs. (64) and (65),

$$\langle P_{\mathrm{tot}}(t)P_{\mathrm{tot}}(t')\rangle = \sum_{\alpha,\beta=-b}^{b} \langle P_\alpha^{(0)} P_\beta^{(0)}\rangle \times |\cos \Omega_\alpha t|^2 |\cos \Omega_\beta t'|^2. \quad (71)$$

We find next the correlation function $\langle P_\alpha^{(0)} P_\beta^{(0)}\rangle$.

5.1. Hanbury–Brown twiss relation for the coherent states

We have found in the last section that the coherent-mode radiation power $\langle P_\alpha \rangle$ is essentially equal to the coherent-mode shot noise power $\mathscr{G}_\alpha^{(1)}$. We relate now the correlation function $\langle P_\alpha P_\beta \rangle$ to $\mathscr{G}_\alpha^{(1)}$ and $\mathscr{G}_\beta^{(1)}$. From Eqs. (65) and (66), we have

$$\langle P_\alpha(t)\, P_\beta(t')\rangle = \langle P_\alpha^{(0)} P_\beta^{(0)}\rangle |\cos \Omega_\alpha t|^2\, |\cos \Omega_\beta t'|^2 \quad (72)$$

and

$$\langle P_\alpha^{(0)} P_\beta^{(0)}\rangle = (4\pi \bar{R})^2 I_{\mathrm{av}}^4 \mathscr{G}_{\alpha\beta}^{(2)} \quad (73)$$

where

$$\mathscr{G}_{\alpha\beta}^{(2)} \equiv \langle |\bar{\mathscr{F}}_\alpha|^2 |\bar{\mathscr{F}}_\beta|^2 \rangle. \quad (74)$$

The quantity $\mathscr{G}_{\alpha\beta}^{(2)}$ has been calculated in the appendix. Using Eq. (A.16), Eq. (73) can be written as

$$\langle P_\alpha^{(0)}\, P_\beta^{(0)}\rangle = (4\pi \bar{R})^2\, I_{\mathrm{av}}^4 (1 + \delta_{\alpha\beta}) \mathscr{G}_\alpha^{(1)} \mathscr{G}_\beta^{(1)}. \quad (75)$$

Combining this equation with Eq. (68), we have

$$\langle P_\alpha^{(0)} P_\beta^{(0)}\rangle = (1 + \delta_{\alpha\beta}) \langle P_\alpha^{(0)}\rangle \langle P_\beta^{(0)}\rangle. \quad (76)$$

This equation together with Eqs. (72) and (65) give

$$\frac{\langle P_\alpha(t) P_\beta(t')\rangle}{\langle P_\alpha(t)\rangle \langle P_\beta(t')\rangle} = (1 + \delta_{\alpha\beta}). \quad (77)$$

$\langle P_\alpha \rangle$ and $\langle P_\beta \rangle$ are not correlated when $\alpha \neq \beta$, but the right-hand sides of the last equation becomes 2 instead of 1 when $\alpha = \beta$. This means that the coherent states are first order coherent but not second order coherent. Note the resemblance of the last two equations to the well-known Hanbury–Brown Twiss relation for the chaotic photons [9,10]. It is significant that the same self-bunching relation applies to the coherent states.

5.2. A flat bunched beam

We apply here the above results to a bunched but flat beam. A flat bunched beam with a bunch length L is defined by

$$\rho(\phi) = \begin{cases} 1/L & \text{if } -L/2 \le \phi < L/2 \\ 0 & \text{otherwise} \end{cases} \quad (78)$$

For the remainder of this section, we shall not distinguish between $\rho^{(b)}(\phi)$ and $\rho(\phi)$.

Recall that the ring circumference $-\pi \le \phi < \pi$ is occupied by $2b+1$ coherent modes. However, if $L < \pi$, only part of the $2b+1$ coherent modes overlap the bunched beam. Therefore, it is convenient to define a parameter "a" by

$$\frac{2a+1}{2b+1} = \frac{L}{2\pi}. \quad (79)$$

Then, for such a bunch,

$$\rho_\alpha^{(b)}(\phi) = \begin{cases} 1/L & \text{if } |\alpha| \le a \\ 0 & \text{if } |\alpha| > a. \end{cases} \quad (80)$$

By a routine combination of the above results, we obtain for such a flat bunched beam

$$\frac{\langle P_{\mathrm{tot}}(t) P_{\mathrm{tot}}(t')\rangle}{\langle P_{\mathrm{tot}}(t)\rangle \langle P_{\mathrm{tot}}(t')\rangle} = 1 + \frac{1}{2a+1}. \quad (81)$$

6. Statistical properties of the coherent-mode shot noise

We have seen in the last section that the Hanbury–Brown Twiss self-bunching relation applies to the coherent state α. This leads us to suspect that the coherent-mode shot noise $\bar{\mathscr{F}}_\alpha$ is a complex Gaussian random variable (CGRV). We study this issue in this section. Start from the definition (34b); it can be rewritten as

$$\bar{\mathscr{F}}_\alpha = \frac{1}{\sqrt{2\pi(2b+1)}} \sum_{\nu=-b}^{b} \exp(-\mathrm{i}\nu\phi_\alpha) \mathscr{H}_\nu \quad (82)$$

where

$$\mathscr{H}_\nu = \int_{-\pi}^{\pi} \mathrm{d}\phi \exp[-\mathrm{i}(n_\mathrm{o} - \nu)\phi] F(\phi). \quad (83)$$

Note that $\mathscr{H}_\nu$ is a random variable since it depends on ϕ_j's. It will be shown below that it is a complex

Gaussian random variable. This in turn implies that $\bar{\bar{\mathscr{F}}}_\alpha$ is a CGRV since a sum of CGRVs is itself a CGRV.

Let us first consider the case when $n_o - v$ is an even integer, namely $n_o - v = 2\tilde{n}$ with integer $\tilde{n}$. Using the wavelength $\lambda \equiv 2\pi(n_o - v)$, Eq. (83) can be written as

$$\mathscr{H}_v = \frac{1}{N}\sum_{m=-\tilde{n}}^{\tilde{n}-1} \mathscr{H}_{v,m} \tag{84}$$

where

$$\mathscr{H}_{v,m} = N\int_{m\lambda}^{(m+1)\lambda} \mathrm{d}\phi \exp(-2\mathrm{i}\tilde{n}\phi)F(\phi). \tag{85}$$

Using Eq. (11), the above equation becomes

$$\mathscr{H}_{v,m} = \sum_{\phi_j} \exp(-2\mathrm{i}\tilde{n}\phi_j) \tag{86}$$

where the summation is over the ϕ_j's which lie in the region of the integration $[m\lambda,(m+1)\lambda]$ of the preceding equation. When ϕ_j varies in this interval, the exponent $2\tilde{n}\phi_j$ of Eq. (86) varies by 2π. The statistical property of $\mathscr{H}_{v,m}$ can be inferred readily from these results. If the electron beam under consideration is a coasting beam, then from $\rho(\phi) = 1/2\pi = \text{constant}$, the random variable ϕ_j's are statistically distributed uniformly in this interval. Therefore [5], $\mathscr{H}_{v,m}$ is a CGRV. Even though the statistical distribution of ϕ_j is no longer a priori uniform if the electron beam is a bunched beam with bunch length σ, the condition (1) of the first section ensures us that it is uniform to the leading order of λ_o/σ, and $\mathscr{H}_{v,m}$ is a CGRV to the same order. This conclusion together with Eq. (84) implies that $\mathscr{H}_v$ is a CGRV.

Having studied the statistical property of $\mathscr{H}_v$ when $n_o - v$ is an even integer, let us turn to the case when $n_o - v$ is an odd integer, namely $n_o - v = 2\tilde{n} + 1$ with integer $\tilde{n}$. For this case

$$\mathscr{H}_v = \frac{1}{N}\sum_{m=-\tilde{n}}^{\tilde{n}} \mathscr{H}_{v,m} \tag{87}$$

where

$$\mathscr{H}_{v,m} = N\int_{(m-1/2)\lambda}^{(m+1/2)\lambda} \mathrm{d}\phi \exp(-\mathrm{i}(2\tilde{n}+1)\phi)F(\phi). \tag{88}$$

Using Eq. (11), the above equation becomes

$$\mathscr{H}_{v,m} = \sum_{\phi_j} \exp(-\mathrm{i}(2\tilde{n}+1)\phi_j) \tag{89}$$

where the summation is over $\phi_j \in [(m-1/2)\lambda, (m+1/2)\lambda]$. An argument similar to the one we used for the even $n_o - v$ case leads us to the conclusion that $\mathscr{H}_v$ is also a CGRV when $n_o - v$ is odd. That $\mathscr{H}_v$ is a CGRV for all v together with Eq. (82) leads to the desired proof that $\bar{\bar{\mathscr{F}}}_\alpha$ is a CGRV. If the electron beam were uniformly distributed around the ring, a so-called coasting beam, the statistical properties of the electron beam would be stationary and the $\bar{\bar{\mathscr{F}}}_\alpha$ would be a complex Gaussian random variables. In this section, we have reached the important conclusion that in spite of the fact that the bunched electron beam is not "stationary" in the global sense $\phi \in [-\pi,\pi]$, the bunched beam can be considered as "stationary" to order λ_o/n_o if condition (1) is satisfied, this in essence is the key assumption in our model of the microwave instability.

From Eqs. (17), (21) and (34a), we have that $\langle \bar{\bar{\mathscr{F}}}_\alpha \rangle = \langle \bar{\bar{\mathscr{F}}}^*_\alpha \rangle = 0$. It is well known that the probability density function for a complex Gaussian random variable with zero mean is given by a Gaussian distribution that in this case becomes

$$\mathscr{P}(\mathrm{Re}\,\bar{\bar{\mathscr{F}}}_\alpha, \mathrm{Im}\,\bar{\bar{\mathscr{F}}}_\alpha) = \frac{N}{\pi\rho_\alpha^{(b)}}\exp\left[\frac{-|\bar{\bar{\mathscr{F}}}_\alpha|^2 N}{\rho_\alpha^{(b)}}\right]. \tag{90}$$

The moments of the distribution yield,

$$\langle |\bar{\bar{\mathscr{F}}}_\alpha|^{2k} \rangle = k!\langle |\bar{\bar{\mathscr{F}}}_\alpha|^2 \rangle^k = k!\left(\frac{\rho_\alpha^{(b)}}{N}\right)^k. \tag{91}$$

It can be seen that these moments are consistent with the results in Eq. (36) for $k = 1$ and Eq. (75) for $k = 2$ with $\alpha = \beta$.

7. Wigner distribution function

The Wigner distribution function can be introduced to yield a quasi- probability density in both the angular position (ϕ) and its Fourier conjugate (v) [11,12]. For a stochastic process the Wigner distribution function is defined in terms of the first-order correlation function [12],

$$W(\phi,v,t)=\int_{-\pi}^{\pi}\mathscr{J}^{(1)}(\phi+\xi/2,\phi-\xi/2,t)\mathrm{e}^{-\mathrm{i}v\xi}\,\mathrm{d}\xi. \tag{92}$$

Substituting from Eq. (61) the Wigner function can be written as a sum over the coherent modes

$$W(\phi,v,t)=\frac{(eN\omega_0)^2}{N}\sum_{\alpha=-b}^{b}\rho_\alpha^{(b)}|\cos\Omega_\alpha t|^2 W_\alpha(\phi,v) \tag{93}$$

where

$$W_\alpha(\phi,v)=\int_{-\pi}^{\pi}\Gamma_\alpha(\phi+\xi/2)\Gamma_\alpha(\phi-\xi/2)\mathrm{e}^{-\mathrm{i}v\xi}\,\mathrm{d}\xi. \tag{94}$$

Note that all the time dependence is contained in $|\cos\Omega_\alpha t|^2$, the expression in Eq. (94) is independent of time. This expression can be simplified by using Eq. (3a) for the definition of $\Gamma_\alpha(\phi)$ and carrying out the subsequent cumbersome, but straightforward, integration over ξ:

$$W_\alpha(\phi,v)=\frac{1}{(2b+1)\sin[(\phi-\phi_\alpha)]}\times\left(\begin{array}{l}\sin[(2b-2|v|+1)(\phi-\phi_\alpha)]\\ +\frac{4}{\pi}\sum_{k=0}^{b-1}\frac{(-1)^{k-v}(1+2k)}{(1+2k)^2-4v^2}\sin[(b-k)2(\phi-\phi_\alpha)]\end{array}\right) \tag{95}$$

for $-b\le v\le b$ and $W_\alpha(\phi,v)=0$ otherwise.

In Fig. 1 we display a plot of $W_\alpha(\phi,v)$ for $\alpha=0$ and $b=5$. As expected, $W_0(\phi,v)$ peaks in the neighborhood of $\phi=\phi_\alpha=0$ and is non-zero only in the range $-5\le v\le 5$.

For the special case of a coasting beam, where all the eigenvalues are the same, all the summations in Eq. (93) can be done and the Wigner distribution function takes a particularly simple form

$$W(\phi,v,t)=\begin{cases}\frac{(eN\omega_0)^2}{2\pi N}|\cos\Omega t|^2 & -b\le v\le b\\ 0 & \text{otherwise.}\end{cases} \tag{96}$$

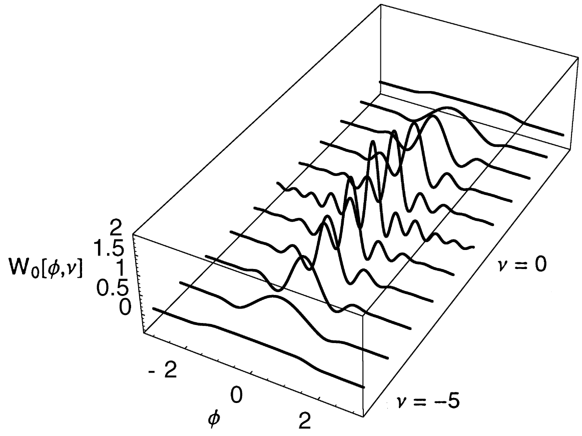

Fig. 1. Plot of $W_\alpha(\phi,v)$ as given in Eq. (95) with $\alpha=0$ and $b=5$.

From the generalized Wigner distribution function defined in Eq. (93) we can obtain the so-called marginal distribution functions

$$\Phi(\phi,t)\equiv\sum_{v=-b}^{b}W(\phi,v,t) \tag{97a}$$

$$=2\pi\mathscr{J}^{(1)}(\phi,\phi,t) \tag{97b}$$

$$=2\pi\frac{(eN\omega_{\mathrm{o}})^2}{N}\sum_{\alpha=-b}^{b}\rho_\alpha^{(b)}|\cos\Omega_\alpha t|^2\Gamma_\alpha^2(\phi) \tag{97c}$$

and

$$\Psi(v,t)=\int_{-\pi}^{\pi}W(\phi,v,t)\,\mathrm{d}\phi \tag{98a}$$

$$=\begin{cases}\frac{2\pi}{2b+1}\frac{(eN\omega_{\mathrm{o}})^2}{N}\sum_{\alpha=-b}^{b}\rho_\alpha^{(b)}|\cos\Omega_\alpha t|^2 & -b\le v\le b\\ 0 & \text{otherwise.}\end{cases} \tag{98b}$$

The marginal distribution $\Phi(\phi,t)$ can be related to the ensemble averaged power given in Eq. (69) as follows:

$$\langle P(\phi,t)\rangle=\frac{\bar{R}}{2\pi^2}\Phi(\phi,t). \tag{99}$$

If we integrate Eq. (97) over ϕ or sum (98) over v we obtain the normalization for the Wigner distribution function,

$$\aleph(t) \equiv \int_{-\pi}^{\pi} \Phi(\phi, t)\,\mathrm{d}\phi = \sum_{v=-b}^{b} \Psi(v, t)$$

$$= 2\pi\frac{(eN\omega_o)^2}{N} \sum_{\alpha=-b}^{b} \rho_\alpha^{(b)}|\cos\Omega_\alpha t|^2. \tag{100}$$

The normalization for the Wigner distribution function $\aleph(t)$ can be related to the ensemble averaged total power given in Eq. (70) as follows:

$$\langle P_{\mathrm{tot}}(t)\rangle = \frac{\bar{R}}{2\pi^2}\aleph(t). \tag{101}$$

8. Entropy

Once again paralleling the analysis for coherent light we can define an entropy-like quantity as follows [13]:

$$H(t) \equiv -\sum_\alpha \wp_\alpha(t)\log\wp_\alpha(t) \tag{102}$$

where

$$\wp_\alpha(t) \equiv \frac{\rho_\alpha^{(b)}|\cos\Omega_\alpha t|^2}{\sum_\beta \rho_\beta^{(b)}|\cos\Omega_\beta t|^2}. \tag{103}$$

Physically, $\wp_\alpha(t)$, is the relative probability that a given coherent mode $\Gamma_\alpha(\phi)$ is excited.

It is of interest to consider what is the form of the $\wp_\alpha(0)$ which maximizes and minimizes the entropy subject to the constraint that $\sum_\alpha \wp_\alpha(0) = 1$. It is straightforward to prove that, $H_{\min} = 0$ if there is a single coherent mode present on the electron beam as there is no uncertainty as to which mode this system is in. In contrast, $H_{\max}(0) = \log(2b+1)$, if the electron beam is uniform, $\rho_\alpha^{(b)} = 1/2\pi$ for all α; the entropy is a maximum as the state of the electron beam is most uncertain. In both of these cases the entropy is independent of time. This is in contrast to a Gaussian electron bunch, where many modes are initially excited; however, the entropy will decrease toward zero with time as the mode with the largest growth rate tends to dominate. It can be seen that the entropy is the logarithm of the number of coherent modes excited on the beam. In this sense the entropy is also a measure of the "coherence" of the instability, lower entropy implies greater coherence.

9. Conclusion

We characterized the microwave instability by a model in which the initial electron beam is assumed to be shot noise which interacts with a constant bandpass impedance in a circular accelerator with periodic boundary conditions. Our analysis applies to the case of a bunched electron beam provided that the wavelength of the disturbance on the beam, the wakelength associated with the model impedance and the electron bunch length satisfy the constraint given in Eq. (1).

The model impedance acts as a filter which bandlimits the initial shot noise. Once the problem is restricted to the space of band-limited functions one can use the Shannon sampling theorem to expand all relevant quantities in terms of periodic band-limited interpolation functions, which we have called coherent states, $\Gamma_\alpha(\phi)$. For our case, of periodic boundary conditions on $[-\pi,\pi]$, there are a finite number of coherent states equal to $2b+1$, which is determined by the frequency spread of the bandpass impedance. We have shown that the Vlasov–Maxwell equations can be separated in space and time. The spatial portion reduces to a Karhunen–Loéve eigenvalue equation of band-limited noise. The associated spatial eigenfunctions are our coherent states and the eigenvalues are related to the localized density of the electron beam. The temporal portion reduces to simple harmonic motion with eigenvalues determined by the impedance and the localized density of the electron beam.

The first- and second-order spatial correlation functions of the electron density where calculated. The initial shot noise is uncorrelated, meaning that the first-order correlation function is characterized by a delta function, $\delta(\phi - \phi')$. When the initial shot noise "passes through the impedance" it introduces a finite correlation length equal to the wakelength. We have shown that the first-order correlation function is diagonalized and is expressible as a sum over our coherent state basis functions. Our correlation function bears a striking resemblance to the correlation function used in the theory of partial coherence in optics [13]. Borrowing from this theory we introduced an entropy like quantity as a measure of the coherence of the instability.

The first-order correlation function of the electron density can be used to obtain a Wigner distribution function. The power in the radiation can be written in terms of the marginal distribution of this Wigner distribution function.

In our consideration of the second-order correlation function we obtained a relation similar to the familiar Hanbury–Brown Twiss condition for the coherent states. We find that the second-order correlation function can simply be written in terms of the first-order correlation function.

We have shown the coherent states, $\Gamma_\alpha(\phi)$, to be an important tool for characterizing both the mathematical and physical aspects of the microwave instability for our assumed model.

Appendix A. The second-order correlation function of the shot noise

The second-order correlation function of the shot noise F is defined by

$$G^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) \equiv \langle F(\phi_1)F(\phi_2)F(\phi_3)F(\phi_4)\rangle. \tag{A.1}$$

The performance of the averaging $\langle \dots \rangle$ above according to the rule specified in Section 2.1 leads to

$$G^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \sum_{j=1}^{5} G_j^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) \tag{A.2}$$

where

$$G_1^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{1}{N^3}[\delta(-\phi_3+\phi_1)\delta(-\phi_4+\phi_1)\delta(\phi_1-\phi_2)\rho^{(b)}(\phi_1)] \tag{A.3}$$

$$G_2^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{(N-3)(N-2)(N-1)}{N^3}\rho^{(b)}(\phi_3)\times\rho^{(b)}(\phi_4)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2) \tag{A.4}$$

$$G_3^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{(N-1)}{N^3}[\delta(\phi_1-\phi_4)\delta(\phi_3-\phi_2)\rho^{(b)}(\phi_3)\times\rho^{(b)}(\phi_1) + \delta(\phi_3-\phi_4)\delta(\phi_1-\phi_2)\rho^{(b)}(\phi_3)\times\rho^{(b)}(\phi_1) + \delta(\phi_1-\phi_3)\delta(+\phi_2-\phi_4)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2)] \tag{A.5}$$

$$G_4^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{(N-1)}{N^3}[\delta(\phi_4-\phi_1)\delta(\phi_1-\phi_2)\rho^{(b)}(\phi_3)\rho^{(b)}(\phi_1) + \delta(\phi_3-\phi_1)\delta(\phi_1-\phi_2)\rho^{(b)}(\phi_4)\rho^{(b)}(\phi_1) + \delta(\phi_3-\phi_1)\delta(\phi_4-\phi_1)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2) + \delta(\phi_3-\phi_2)\delta(\phi_4-\phi_2)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2)] \tag{A.6}$$

and

$$G_5^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{(N-2)(N-1)}{N^3}[\delta(\phi_1-\phi_2)\rho^{(b)}(\phi_3)\rho^{(b)}(\phi_4)\times\rho^{(b)}(\phi_1) + \delta(\phi_3-\phi_4)\rho^{(b)}(\phi_3)\rho^{(b)}(\phi_1)\times\rho^{(b)}(\phi_2) + \delta(\phi_4-\phi_1)\rho^{(b)}(\phi_3)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2) + \delta(\phi_4-\phi_2)\rho^{(b)}(\phi_3)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2) + \delta(\phi_3-\phi_1)\rho^{(b)}(\phi_4)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2) + \delta(\phi_3-\phi_2)\rho^{(b)}(\phi_4)\rho^{(b)}(\phi_1)\rho^{(b)}(\phi_2)]. \tag{A.7}$$

We calculate next the correlation function $\mathscr{G}_{\alpha\beta}^{(2)}$ of the localized shot noise power. It is defined by

$$\mathscr{G}_{\alpha\beta}^{(2)} \equiv \langle|\bar{\mathscr{F}}_\alpha|^2|\bar{\mathscr{F}}_\beta|^2\rangle. \tag{A.8}$$

We have just seen above that the correlation function $G^{(2)}$ consists of five components. This together with the definition of the coherent-mode shot noise $\bar{\mathscr{F}}_\alpha$ given by Eq. (34b) imply that $\mathscr{G}^{(2)}$ also consists of five components

$$\mathscr{G}^{(2)}_{\alpha\beta} = \sum_{j=1}^{5} \mathscr{G}^{(2)}_{j,\alpha\beta} \tag{A.9}$$

where

$$\mathscr{G}^{(2)}_{j,\alpha\beta} = \left[\prod_{k=1}^{4} \int_{-\pi}^{\pi} \mathrm{d}\phi_k\right] G^{(2)}_j(\phi_1,\phi_2,\phi_3,\phi_4)\Gamma_\alpha(\phi_1)\Gamma_\alpha(\phi_2) \times \Gamma_\beta(\phi_3)\Gamma_\beta(\phi_4)\mathrm{e}^{-\mathrm{i}n_o(\phi_1-\phi_2)}\mathrm{e}^{-\mathrm{i}n_o(\phi_3-\phi_4)}. \tag{A.10}$$

Let us first calculate the third component $\mathscr{G}^{(2)}_3$. There are three contributions to $\mathscr{G}^{(2)}_3$ from, respectively, the three terms on the right-hand side of Eq. (A.5). The last term of Eq. (A.5) gives a contribution proportional to

$$\left|\int_{-\pi}^{\pi} \mathrm{d}\phi\, \rho^{(b)}(\phi)\Gamma_\alpha(\phi)\Gamma_\beta(\phi)\exp(-2\mathrm{i}n_o\phi)\right|^2. \tag{A.11}$$

This integral vanishes because (i) $\rho^{(b)}$, Γ_α and Γ_β are all members of $\mathscr{M}_b$, and (ii) $n_o \gg b$. The calculation of the other two contributions gives

$$\mathscr{G}^{(2)}_{3,\alpha\beta} = \frac{N-1}{N^3}[(\rho^{(b)}_\alpha\delta_{\alpha\beta})^2 + (\rho^{(b)}_\alpha)^2(\rho^{(b)}_\beta)^2]. \tag{A.12}$$

Taking the limit $N \to \infty$, and using Eq. (36), the above equation can be written in terms of the average local mode noise power $\mathscr{G}^{(1)}_\alpha$ as

$$\mathscr{G}^{(2)}_{3,\alpha\beta} = (1+\delta_{\alpha\beta})\mathscr{G}^{(1)}_\alpha\mathscr{G}^{(1)}_\beta. \tag{A.13}$$

The other $\mathscr{G}^{(2)}_{j,\alpha\beta}$'s can be calculated in a similar way. We find

$$\mathscr{G}^{(2)}_{2,\alpha\beta} = \mathscr{G}^{(2)}_{4,\alpha\beta} = \mathscr{G}^{(2)}_{5,\alpha\beta} = 0 \tag{A.14}$$

and

$$\mathscr{G}^{(2)}_{2,\alpha\beta} = \frac{1}{N^2}\frac{2b+1}{2\pi}\mathscr{G}^{(1)}_\alpha\delta_{\alpha\beta}. \tag{A.15}$$

$\mathscr{G}^{(2)}_2$ is also negligible since it is smaller than $\mathscr{G}^{(2)}_3$ by a factor of $1/N$. We conclude that of the five components of $\mathscr{G}^{(2)}$, only $\mathscr{G}^{(2)}_3$ is appreciable and

$$\mathscr{G}^{(2)}_{\alpha\beta} \cong \mathscr{G}^{(2)}_{3,\alpha\beta} = (1+\delta_{\alpha\beta})\mathscr{G}^{(1)}_\alpha\mathscr{G}^{(1)}_\beta. \tag{A.16}$$

We have seen in Eq. (29) that under the condition (1), we have effectively

$$G^{(1)}(\phi,\phi') = \frac{1}{N}\rho^{(b)}(\phi)\delta(\phi-\phi'). \tag{A.17}$$

Eq. (A.16) amounts to setting

$$G^{(2)}(\phi_1,\phi_2,\phi_3,\phi_4) = G^{(1)}(\phi_1,\phi_2)G^{(1)}(\phi_1,\phi_2) + G^{(1)}(\phi_1,\phi_3)G^{(1)}(\phi_2,\phi_4) + G^{(1)}(\phi_1,\phi_4)G^{(1)}(\phi_2,\phi_3), \tag{A.18}$$

with $G^{(1)}$ given by (A.17).

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